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**LETTER TO THE EDITOR**

# ‘Can one hear the shape of a drum?’: revisited

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**Abstract**
A famous inverse problem posed by M Kac ‘Can one hear the shape of a drum?’ is concerned with isospectrality of drums or planer billiards, and the first counter example was constructed by Gordon, Webb and Wolpert (1992 *Invent. Math.* **110** 1). Here we present pieces of numerical evidence showing that ‘One can distinguish isospectral drums by measuring the scattering poles of exterior Neumann problems’. This is based on the observation that the Fredholm determinant appearing in the boundary element method admits a factorization into interior and exterior parts.

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A famous question, ‘can one hear the shape of a drum?’, which was posed by Kac is the inverse eigenvalue problem for the Helmholtz equation [1]. The question involves not only the vibrating membrane fixed with a rigid frame, but also various physical systems obeying the Helmholtz equation. In particular, quantum mechanical eigenvalue problems in bounded two-dimensional domains, which are called quantum billiard problems, attract much attention in conjunction with quantum manifestation of classical chaos. If the answer to the question is yes, that is, one can hear the shape of a drum, then the set of quantum energy levels is enough to specify the shape of a billiard table, and otherwise some information is missing. Counter examples of similar questions had already been given prior to Kac’s; Milnor found two flat 16-dimensional tori that are not congruent but are nevertheless isospectral [2]. Also after Kac’s, it was also proved that there exist non-isometric but isospectral pairs on certain Riemannian manifolds [3–5], and counter examples in the same sense were presented for quantum graphs [6]. They employed Sunada’s theorem, which gives a sufficient condition that pairs of Riemannian manifolds become isospectral [3]. However, Kac’s original question is concerned with isospectrality of planar domains. The first counter example for Kac’s question was constructed by Gordon, Webb and Wolpert [7], who gave a concrete example of an isospectral but non-congruent pair of domains, and then 17 families of isospectral pairs of domains were found [8]. On the other hand, if we limit ourselves to a certain class of plane domains, it is also known that the shape can be determined by its spectrum. In order to see this aspect of the isospectral problem, we refer to the recent review by Zelditch [9].

As mentioned, the existence of isospectral pairs means that a set of eigenenergies is not sufficient to determine the shape of the billiard boundary. This immediately invokes a question; what is needed, if possible, to distinguish isospectral pairs? One of the meaningful settings would be to limit ourselves to the situation where one can only know physically observable quantities, just as eigenmodes of a drum can be observed or heard as sound.

Inside–outside duality of quantum billiard problems, which has been investigated in several different ways [10–14], may provide us with a clue to approach this question. Though there are several versions with respect to the boundary conditions imposed in each inside and outside problem, and the assertions themselves are different, the results imply that eigenstates for inside billiard tables are related to scattering states for outside problems. In this letter, we will present that the shape of isospectral billiards can indeed be distinguished by observing exterior Neumann scattering problems. Our idea originates from the fact that the Fredholm determinant, which appears in the boundary integral method (BIM), admits a factorization into interior and exterior contributions [13].

To be more precise, let us formulate the BIM in terms of the Fredholm theory [13, 15, 16]. Our present concern is the Helmholtz equation for a two-dimensional bounded domain $\Omega$:

$$\Delta\Psi(r) + k^2\Psi(r) = 0 \qquad (r \in \Omega), \tag{1}$$

with the homogeneous Dirichlet boundary condition:

$$\Psi(r) = 0 \qquad (r \in \partial\Omega). \tag{2}$$

Writing a solution of (1) as the double-layer potential with a density $\rho$:

$$\Psi(r) = \int_{\partial\Omega} \frac{\partial G_0(r, r(s); k)}{\partial\nu_s} \rho(s)\,\mathrm{d}s, \tag{3}$$

one can express the boundary condition (2) as an integral equation:

$$\rho(t) - \int_{\partial\Omega} K(t, s; k)\rho(s)\,\mathrm{d}s = 0, \tag{4}$$

$$K(t, s; k) := -2\frac{\partial G_0(r(t), r(s); k)}{\partial\nu_s}, \tag{5}$$

where $\nu_s$ denotes the outer unit vector at an arc length $s$ measured along the boundary $\Omega$. $G_0(r, r'; k)$ is a Green function for the free two-dimensional space, which we take

$$G_0(r, r'; k) := -\frac{\mathrm{i}}{4} H_0^{(1)}(k|r - r'|). \tag{6}$$

If we limit ourselves to domains whose boundaries are of class $C^2$, the integral kernel $K$ can be continuously extended to the whole of $\partial\Omega \times \partial\Omega$ by setting its diagonal as $K(t, t; k) = -\kappa(t)/2\pi$, where $\kappa(t)$ denotes the curvature of the boundary at the point $r(t)$. Then one can apply the Fredholm theory to the integral equation (4). As a result, the eigenvalues of equations (1), (2) are completely characterized by the zeros of the Fredholm determinant:

$$d(k) := 1 + \sum_{j=1}^{\infty} d_j(k), \tag{7}$$

where

$$d_j(k) := \frac{(-1)^j}{j!} \int_{\partial\Omega} \mathrm{d}s_1 \cdots \int_{\partial\Omega} \mathrm{d}s_j \begin{vmatrix} K(s_1, s_1; k) & \cdots & K(s_1, s_j; k) \\ \vdots & \ddots & \vdots \\ K(s_n, s_1; k) & \cdots & K(s_n, s_j; k) \end{vmatrix}. \tag{8}$$

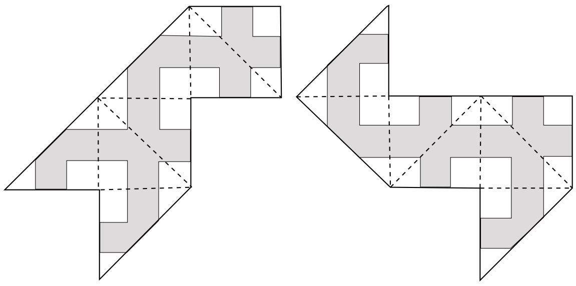

**Figure 1.** Isospectral pairs of domains whose isospectrality is checked by the same transplantation of eigenfunctions.

It was shown in [13] that the Fredholm determinant admits decomposition into the interior and exterior contributions. More precisely, $d(E)$ is factorized as

$$d(E) = d(0)d_{\rm int}(E)d_{\rm ext}(E), \qquad (9)$$

where $E = \hbar^2k^2/2m$. Here the interior term $d_{\rm int}(E)$ gives Hadamard's factorization:

$$d_{\rm int}(E) = {\rm e}^{{\rm i}\frac{m|\Omega|E}{2\hbar^2}} \left(\frac{m|\partial\Omega|E}{2\hbar^2}\right)^{-\frac{m|\Omega|E}{2\pi\hbar^2}} {\rm e}^{-\frac{m|\Omega|\gamma' E}{\pi\hbar^2}} \prod_{n=1}^{\infty}\left(1-\frac{E}{E_n}\right){\rm e}^{\frac{E}{E_n}}, \qquad (10)$$

and $\{E_n\}_{n=1}^{\infty}$ denote the eigenenergies of the interior Dirichlet problem. $|\partial\Omega|$ and $|\Omega|$ represent the perimeter and the area of the domain $\Omega$, respectively. $\gamma'$ is a constant determined by the geometry of $\Omega$. Furthermore, the analytic extension of the exterior term $d_{\rm ext}(E)$ to the second Riemann sheet is connected to the $S$-matrix for the exterior Neumann scattering $S$ by the relation:

$$d_{\rm ext}^{II}(E) = {\rm e}^{-{\rm i}\frac{m|\Omega|E}{\hbar^2}} \frac{d_{\rm ext}(E)}{\det S^{II}(E)}. \qquad (11)$$

Here $S^{II}$ denotes the analytic extension of the on-shell $S$-matrix $S$. Note that the second Riemann sheet of $E$ corresponds to the lower half plane of $k$. This factorization tells us that the Fredholm determinant $d(k)$, from which eigenenergies of the interior Dirichlet problem are obtained, carries at the same time information on the cross section of the exterior Neumann scattering problems as its zeros. Alternatively stated, eigenenergies are not sufficient information to specify the Fredholm determinant $d(k)$, which is uniquely determined if the shape of the billiard boundary is given. This is suggestive for our problem addressed above: even if the shape of a drum cannot be heard, one may distinguish isospectral drums by measuring sound scattered by the drums [13].

We should remark that, in its strict sense, the argument based on the Fredholm theory should be applied only for domains encircled by smooth curves. On the other hand, all the isospectral pairs of planar billiards found so far do not satisfy this condition, and as explained below this fact may be inevitable for isospectrality problems of planar billiards. Figure 1 shows a pair of isospectral billiard tables, which was obtained as quotient orbifolds of a common manifold by Gordon *et al* [7]. Although isospectrality of these domains was originally proved with the aid of Sunada's theorem, it can be more easily checked by the method of *transplantation* of eigenfunctions [8]. The procedure of transplantation is just to cut an eigenfunction $f$ on a domain $\Omega_1$ into different pieces $f_i$, a function defined on each building block, and to paste them on the other domain $\Omega_2$. If the pasted function $g$ satisfies a proper boundary condition $g = 0$ on $\partial\Omega_2$, and also it is smoothly connected on all reflecting

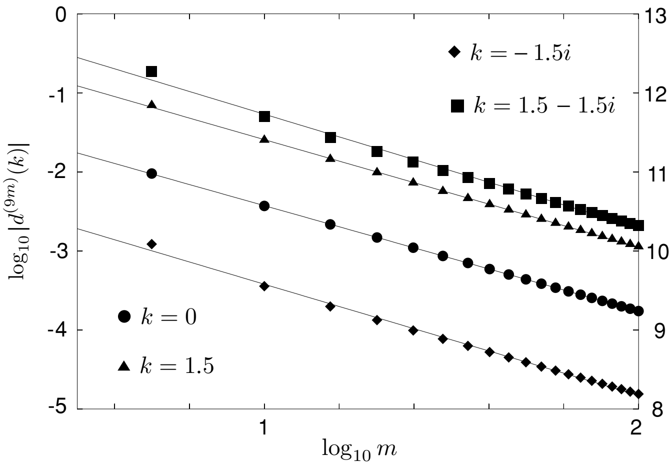


**Figure 2.** Convergence of discretized determinant $d^{(n)}(k)$. The shape of the billiard table is the left-hand side unfolded domain shown in figure 1. Here $m$ denotes the number of boundary points lying on each edge of the building block. The total number of discretized points is given as $n = 9m$ where 9 is the number of sides of unfolded domain.

segments, we can say that $g$ is an eigenfunction on $\Omega_2$. This procedure can be formulated as a simple transformation between its adjacency matrices [17]. All the transplantable pairs enumerated so far coincide with the examples obtained by group theoretical arguments [17]. An important fact to be stressed here is that finding isospectral pairs is therefore reduced to a problem of combinatorics. All the counter examples of Kac's problem ever known are, to authors' knowledge, domains which are obtained by gluing several copies of a building block, and its isospectrality is checked by the transplantation method mentioned above. Such domains necessarily have corners.

When a domain has a corner, the corresponding integral kernel $K(t, s; k)$ appearing in BIM behaves as $|r(s) - r(t)|^{-1}$ around the corner. Thus, the integral equation (4) becomes singular, and convergence of the infinite series (7) cannot be guaranteed. As far as the interior problem is concerned, such a difficulty can be overcome by employing the wedge kernel [18], but it is not sufficient in the present issue. So, it is not obvious whether or not the above factorization formula holds as it stands for isospectral billiards. Nevertheless, the question posed above still makes sense since eigenvalues of the interior Dirichlet problem and cross sections or poles of $S$-matrix of Neumann scattering problems are well-defined objects for unfolded domains, and they are still the solutions of the singular integral equation (4) [19, 20].

Here we will cope with this issue in the following: first, recall the discretized determinant,

$$d^{(n)}(k) = \det(\delta_{ij} - w_j K_{ij}(k)), \tag{12}$$

which is the determinant of the discretized version of our integral equation (4),

$$\rho_i - \sum_{j=1}^{n} w_j K_{ij}(k)\rho_j = 0, \tag{13}$$

where $K_{ij}(k) := K(t_i, t_j; k)$. Here $n$ is the number of discretized points on the boundary and $w_j$ denote the weight factor for each boundary point $r(t_j)$. In the case of the billiard table with a smooth boundary, it was shown that $\lim_{n\to\infty} d^{(n)}(k) = d(k)$ by taking $t_j = |\partial\Omega| j/n$ and $w_j = |\partial\Omega|/n$.

On the other hand, if the boundary has corners, $d^{(n)}(k)$ tends to zero. Indeed as observed in figure 2, even if we use Gauss–Legendre quadrature, which is a proper discretization to give

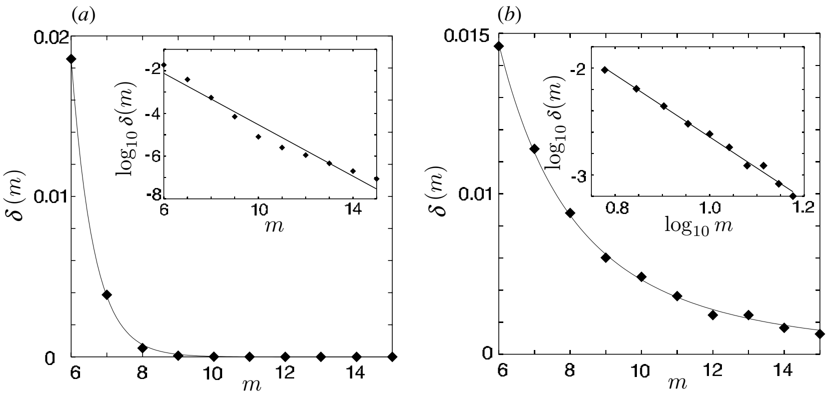


**Figure 3.** Convergence of zeros of discretized determinant for the left-hand side billiard in figure 1. Here $\delta(m)$ is the distance between a zero of $\tilde{d}_{9m}(k)$ and one of $\tilde{d}_{9(m-1)}(k)$ in the vicinity of (*a*) $k = 5.0$ and (*b*) $k = 1.2 - 0.28\mathrm{i}$.

a correct line integral [21], $d^{(n)}(k)$ tends to zero for all $k$ as a function of $n$. Although its ratio is algebraically slow, this makes $d^{(n)}(k)$ meaningless as $n \to \infty$. However, it can also be found in figure 2 that the exponent of algebraic decrease of $d^{(n)}(k)$ does not depend on $k$, implying that $d^{(n)}(k)$ tends to zero uniformly. We can use this fact to introduce a regularization factor for $d^{(n)}(k)$. That is, instead of computing $d^{(n)}(k)$, we evaluate

$$\tilde{d}^{(n)}(k) := \frac{d^{(n)}(k)}{d^{(n)}(0)}. \tag{14}$$

In fact, as demonstrated in figure 3, we can see that zeros of $\tilde{d}_n(k)$, together with a proper quadrature, show well convergent properties. In figure 3, we have plotted the distance between a zero of $\tilde{d}_n(k)$ and one of $\tilde{d}_{n-1}(k)$ as a function of $n$. In the case of eigenvalues of the interior problem, i.e., zeros of $\tilde{d}_n(k)$ very close to the real $k$ axis, the location of zeros converges exponentially, whereas zeros on the lower half plane converge algebraically slowly. However, in either case, the precision thus determined is enough to distinguish the location of zeros unambiguously.

As will be fully explained in [22], the regularization procedure (14) can be justified rigorously. This is based on the decomposition of integral operator $K(k)$, whose integral kernel is $K(s, t; k)$, into a singular part $K_s$ and a compact one $K_r(k)$ such that $|K_s|$ is less than unity [23]. Since the $I - K_s$ has a bounded inverse provided by Neumann series, the singular integral equation can be converted into a non-singular one:

$$\rho - (I - K_s)^{-1} K_r(k) \rho = 0. \tag{15}$$

Denoting the Fredholm determinant for the integral equation (15) by $D(k)$, one can easily prove

$$\lim_{n\to\infty} \tilde{d}_n(k) = \frac{D(k)}{D(0)}. \tag{16}$$

On the basis of these arguments to justify the BIM for billiard tables with corners, we numerically compute the location of zeros of Fredholm determinants and the resonance counting function for four types of isospectral billiards. Since isolated information will not be so helpful to further understand the inside and outside relation, we examine several types of isospectral pairs. The first one is a pair of unfolded domains whose fundamental building block is a right-angled triangle. As mentioned already, we may replace a fundamental building block

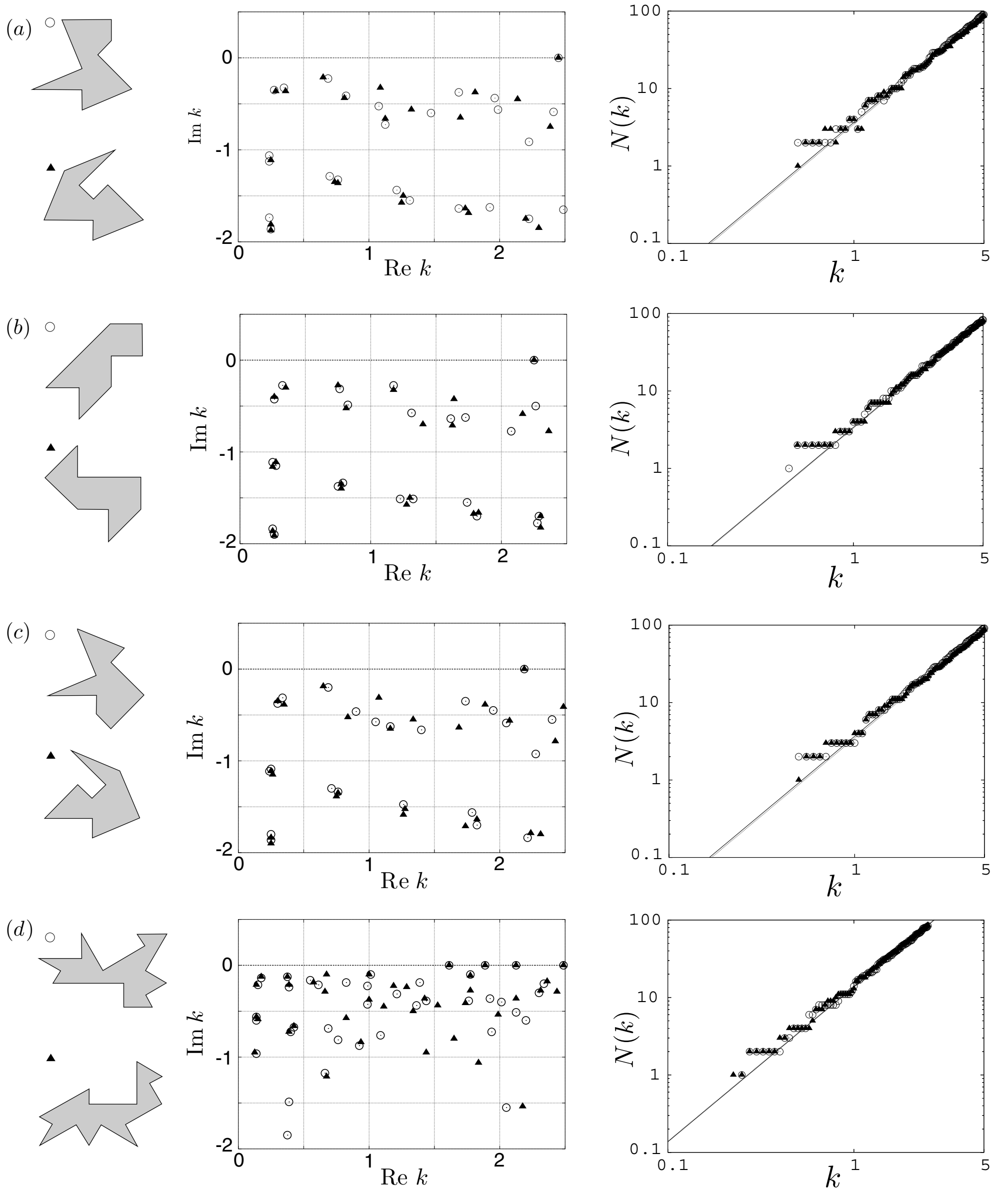


**Figure 4.** Left: distribution of zeros of the discretized determinant $\tilde{d}_n(k)$ for isospectral billiards. The local minima of $|\tilde{d}_n(k)|$ are marked on the complex $k$ plane. The number of discretized boundary points are 30 for each edge. The error of location of zeros is smaller than the size of the circles plotted. Right: log–log plot of the resonance counting function $N(k)$, which is the number of resonances in the sector $\{k' \in \mathbf{C} \,||k'| < k, -\pi/2 < \arg(k') < -\pi/100\}$. The resonances are counted in every annulus with the width 0.01. We also plot the mean resonance counting function $Ck^2$, (*a*) $C = 3.66$ for the upper domain and $C = 3.46$ for the lower, (*b*) $C = 3.38$ for the upper and $C = 3.31$ for the lower, (*c*) $C = 3.66$ for the upper and $C = 3.46$ for the lower, (*d*) $C = 13.7$ for the upper and $C = 13.9$ for the lower. These coefficients are determined by the nonlinear least-squares Marquardt–Levenberg algorithm.

by an arbitrary shape as long as it has three reflecting sides about which unfolded domains are constructed. So, the second example shown as figure 4(*b*) is a pair of unfolded domains

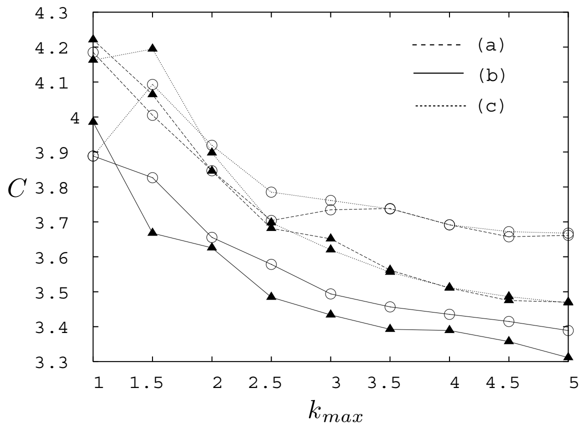


**Figure 5.** (*a*) The optimal coefficient of $k^2$ to fit $N_{\pi/100}(k) = \{k' \in \mathbf{C}\,||k'| < k, -\pi/2 < \arg(k') < -\pi/100\}$ restricted to $0 < k < k_{\max}$. The dashed, solid and dotted lines correspond to the examples (*a*), (*b*) and (*c*) in figure 4, respectively. The circles and triangles denote upper and lower domains in figure 4, respectively.

composed of a different building block. The third one is the case in which the connection rule to give unfolded domains differs from the former two cases. As seen in Buser's table [8] and also in the table of transplantable pairs [17], even if one fixes the number of building blocks, there sometimes exist several different unfolded patterns, all of which give isospectral pairs. In figure 4(*c*) we have tested such a pair. The final one shown in figure 4(*d*) is an example of the homophonic pair. We say the two domains are homophonic if there exists a point in the domain such that the values of all the eigenfunctions are the same. As a result, if one exactly hits this particular point, the sound of two drums is completely the same. Such a pair first appears when the number of unfolded domains is 21 [8].

All the examples examined here show that complex zeros of the Fredholm determinant are different whereas zeros on the real axis are the same; this naturally leads us to the conjecture that an isospectral pair can be distinguished by measuring the poles of *S*-matrix or the cross section of exterior Neumann scattering.

As can also be seen in figure 4, the number of resonances whose absolute value is less than $k$ increases in proportion to $k^2$. The results are consistent with the upper bounds of the resonance counting function for the Dirichlet boundary condition [24], although their results concern that for the Dirichlet boundary condition.

We note that the resonance counting function carries further information; as seen in figure 5, the optimal coefficients $C_{\delta,k}$ of $k^2$ to fit $N_\delta(k)$ differ from each other when we compare isospectral pairs. Here, the resonance counting function in a fixed-angled sector is defined as $N_\delta(k) = \{k' \in \mathbf{C}\,||k'| < k, -\pi/2 < \arg(k') < -\delta\}$. The results look inconsistent with the Weyl-type formula that was shown for exterior non-trapping domains [25]. However, this is not necessarily the case, since $N_\delta(k)$ in figures 4 and 5 plots only a part of the resonances that are located in the sector with the central angle $\delta$. Thus, it is possible that Weyl-type formula is recovered if we sweep the semicircle in the lower half plane. In other words, $N_\delta(k)$ can have $\delta$ dependence.

Another issue we should further examine in relation to it is the asymptotic behaviour of the mean resonance function in the case of trapping domains appearing in figures 4(*a*), (*c*) and (*d*). This is because rigorous estimations provided in [24, 25] are concerned only with non-trapping domains. To the authors' knowledge, there are no proofs claiming that the mean resonance function for trapping domains obeys an analogous Weyl-type formula. So, if this is the case, it would be interesting to see such a discrepancy in the mean resonance counting function in terms of semi-classical approximation of the Fredholm determinant [26]. In any case, the result presented in figure 5 needs further investigations and should be reconsidered in a future publication. Developments of more efficient numerical algorithms are highly desired to this end.

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